MARKOV PROCESSES ON A LOCALLY COMPACT SPACE

BY

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ABSTRACT

Given a positive contraction, P, on $C(X)$ we define the conservative and dissipative parts of P and establish some properties which are analogous to known ones from measure theory (see [3]). We also prove a ratio limit theorem for certain processes.

1. **Definitions and notations.** Let X be a locally compact, σ-compact and perfectly normal space. Let P be an operator on $C(X)$ such that:

(i) P is positive, i.e., if $f \geqq 0$ then $Pf \geqq 0$

(ii) P is a contraction. i.e., $\| P \| \leqq 1$.

(iii) The adjoint operator P^*, that acts on the space of the regular charges, preserves the space of the regular measures, i.e., if λ is a measure on X then $P^*\lambda$ is also a measure.

Such an operator defines a Markov transition probability on (X,Σ) where Σ is the Borel σ-field, in the following form:

$$P(x,A) = (P^*\delta_x)(A) \tag{1.1}$$

where δ_x is the Dirac measure at x. It is clear that $P(x,\cdot)$ is a measure for all x. On the other hand, if f is a continuous function then $\langle P^*\delta_x,f\rangle = Pf(x)$ is also continuous, and the collection $\mathfrak{A} = \{f \mid f \in \mathfrak{B}(X,\Sigma); \langle P^*\delta_x,f\rangle \in \mathscr{B}(X,\Sigma)\}$ is equal to $\mathscr{B}(X,\Sigma)$, the space of the bounded and Σ-measurable functions, because A contains all the continuous functions and is closed under monotonic limits. Hence, if f is measurable then $Pf(x) = \langle P^*\delta_x, f\rangle$ is also measurable. In particular, for every $A \in \Sigma$, $P(\cdot,A)$ is a measurable function. Hence, $P(x,A)$ defined in (1.1) is indeed a Markov transition probability.

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Let $\alpha(x)$ be a function, we shall define the operator:

$$I_\alpha f(x) = \alpha(x)\cdot f(x) \tag{1.2}$$

Particularly if $\alpha(x) = 1_A(x)$, $A \in \Sigma$ we shall denote:

$$I_\alpha = I_A \tag{1.3}$$

2. The conservative and the dissipative parts of the process.

DEFINITION 2.1. A function f is said to be lower semi-continuous (l.s.c.) if there exists an increasing sequence of continuous functions $\{f_n\}$ so that $f_n \nearrow f$.

(2.1) If f_1, f_2 are l.s.c. then $\min(f_1, f_2)$ is l.s.c.

(2.2) If $\{f_n\}$ are l.s.c. then $\lim_{n\to\infty} \max(f_1, \cdots, f_n)$ is l.s.c.

(2.3) If A is an open set then 1_A is l.d.c. If f is l.s.c. then $\{f > a\}$ is open and $\{f \leqq a\}$ is closed.

(2.4) If f is l.s.c. then Pf is l.s.c.

function $f \in B(X,\Sigma)$ will be called sub-variant if $Pf \leqq f$.

The following lemma is Lemma 6 of [5]. We shall give here the proof for completeness.

LEMMA 2.1. *Let $A \in \Sigma$, there exists a function i_A which is minimal with respect to the condition $1_A \leqq i_A \leqq 1$ and $P_{i_A} \leqq i_A$. The function i_A can be represented as follows:*

$$i_A = \sum_{n=0}^{\infty} (I_{A^c}P)^n 1_A \quad (\textit{where } A^c = X - A). \tag{2.5}$$

If A is open then i_A is l.s.c.

Proof. See [3] Chapter III, formulas (3.1) and (3.2). If A is an open set then the function 1_A is l.s.c. Denote $g_N = \sum_{n=0}^{N} (I_A cP)^n 1_A$. If g_N is l.s.c. then also $g_{N+1} = \sum_{n=0}^{N+1} (I_{A^c}P)^n 1_A = 1_A + I_A P g_N = \max(1_A, Pg_N)$ is l.s.c. But $g_N \uparrow i_A$, hence i_A is l.s.c.

$Pi_A(x)$ is the probability that x enters A at least once. The sequence $P^n i_A$ is decreasing, hence the limit $\lim_{n\to\infty} P^n i_A$ exists. $\lim P^n i_A(x)$ is the probability that x enters A infinitely many times.

DEFINITION 2.2. A set A is said to be inessential if $\lim_{n\to\infty} P^n i_A(x) = 0$ for all x.

The union of all open and inessential sets is said to be the dissipative part of the process and will be denoted by D.

Clearly D is an open set, and since X is perfectly normal and σ-compact, there exists a sequence of open and inessential sets $\{D_n\}$ so that

$$D = \bigcup_{n=1}^{\infty} D_n \tag{2.6}$$

DEFINITION 2.3 The conservative part of the process C is

$$C = X - D \tag{2.7}$$

REMARK. It is not difficult to see that the definition of D is equivalent to the definition of W in [5] formula (7).

THEOREM 2.1. *There exists a decomposition of D.*

$$D = \bigcup_{n=1}^{\infty} E_n \cup N \tag{2.8}$$

where E_n are open sets and $\sum_{k=1}^{\infty} P^k 1_{E_n} \in \mathscr{B}(X,\Sigma)$ and N is a set of first category.

Proof. Let us consider the decomposition $D = \bigcup_{n=1}^{\infty} D_n$ of formula (2.6) Define: $\bar{E}_{nj} = \{x \mid P^j i_{D_n}(x) \leqq \frac{1}{2}\}$. $\bar{E}_{nj}$ is closed. We have $\bigcup_{j=1}^{\infty} \bar{E}_{nj} = X$. Denote $E_{nj} = \operatorname{int} \bar{E}_{nj}$. The set $\bar{E}_{nj} - E_{nj}$ is nowhere dense, and the set $N_n = \bigcup_{j=1}^{\infty} (\bar{E}_{nj} - E_{nj})$ s of the first category.

$$D_n = \bigcup_{j=1}^{\infty} (E_{nj} \cap D_n) \cap (N_n \cup D_n).$$

Thus

$$i_{D_n} - P^j i_{D_n} \geqq \tfrac{1}{4} 1_{E_{nj} \cap D_n}.$$

Hence:

$$\sum_{k=1}^{k} P^k 1_{B_{nj} \cap D_n} \leqq 4 \sum_{k=1}^{k} P^k (i_{D_n} - P^j i_{D_n}) \leqq 4j$$

denote $N = \bigcup_{n=1}^{\infty} (N_n \cap D_n)$ this is a set of first category. But

$$D = \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} (E_{nj} \cap D_n) \cup N,$$

so the theorem is proved.

In the rest of this section we shall assume that $D = \varnothing$ i.e. $X = C$.

REMARK. An intersection between an open set A and a closed set B is either nowhere dense or contains an open set, because if $E \subset \bar{A} \cap B$ is open then $E \cap A$ is not empty and $E \cap A \subset A \cap B$.

THEOREM 2.2. *Let A be any open set; then i_A takes only the values 0 or 1 except on a set of first category. The same is true for $P^n i_A$ for each n.*

Proof. Let us denote $N_n = \{Pi_A > 1/n,\ P^n i_A \leqq 1 - 1/n\}$. If the set N_n is not nowhere dense then there exists an open set $B \subset N_n$, because N_n is an intersection of the open set $\{Pi_A > 1/n\}$ and the closed set $\{P^n i_A \leqq 1 - 1/n\}$. By Proposition 10 of [1] B is an inessential set because $\inf_{x \in B} Pi_A(x) > 0$ and $\sup_{x \in B} \lim_{k \to \infty} P^k i_A(x) \leqq 1 - 1/n < 1$ which contradicts the assumption $D = \emptyset$. Hence $P^n i_A$ takes only the values 0 or 1 for each n, except on the set of first category $N = \bigcup_{n=1}^{\infty} N_n$. Let us denote $M_n = \{Pi_A = 0,\ i_A > 1/n\}$. If the set M_n is nowhere dense, then there is an open set $\emptyset \neq B \subset M_n$, because M_n is an intersection of the open set $\{i_A > 1/n\}$ and the closed set $\{Pi_A \leqq 0\}$. We have $n(i_A - Pi_A) \geqq 1_B$. Define $f = \min(1, n \sum_{k=0}^{\infty} P^k(i_A - Pi_A))$. This function satisfies $1_B \leqq f \leqq 1$. By Lemma 2.1 $i_B \leqq f$. Hence

$$\lim_{j \to \infty} P^j{}_{i_B} \leqq \lim_{j \to \infty} P^j f \leqq \lim_{j \to \infty} n \sum_{k=0}^{\infty} P^{k+j}(i_A - P_{i_A}) = 0$$

which contradicts the assumption $D = \emptyset$. Hence $Pi_A(x) = 0$ implies $i_A(x) = 0$ except on the set of first category $M = \bigcup_{n=1}^{\infty} M_n$. It is clear that $Pi_A(x) = 1$ mplies $i_A(x) = 1$, hence i_A takes only the values 0 or 1 except on the set of first category $M \cup N$, and this completes the proof of the theorem.

THEOREM 2.3. *Let A be any open set. Then*

$$\sum_{n=0}^{\infty} P^n 1_A(x) = \{^0_\infty \tag{2.10}$$

We shall first prove the following lemma:

LEMMA 2.2. *Let $x \in X$ and $A \in \Sigma$ then $\sum_{n=1}^{\infty} P^n(x, A) = 0$ if and only if $Pi_A(x) = 0$.*

Proof. (a) If $\sum_{n=1}^{\infty} P^n(x, A) = 0$ then by (2.5) we have:

$$Pi_A(x) = \sum_{n=0}^{\infty} P(I_A P)^n 1_A(x) \leqq \sum_{n=1}^{\infty} P^n(x, A) = 0$$

(b) If $Pi_A(x) = 0$ then by Lemma 2.1, we have

$$P^n(x,A) \leqq P^n i_A(x) \leqq P i_A(x) = 0$$

hence $\sum_{n=1}^{\infty} P^n(x,A) = 0$.

Proof. of Theorem 2.3 (a) By Lemma 2.2 we have $i_A(x) = 0 \Rightarrow \sum_{n=0}^{\infty} P^n 1_A(x) = 0$.

(b) By Theorem 2.2 $\lim_{k\to\infty} P^k i_A(x)$ takes only the values 0 or 1 except on a set of first category, but:

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} P^{n+k} 1_A(x) \geqq \lim_{k\to\infty} \sum_{n=0}^{\infty} P^k (I_{A^c}P)^n 1_A(x) = 1.$$

Hence, $\lim_{k\to\infty} P^k i_A(x) = 1 \Rightarrow \sum_{n=0}^{\infty} P^n 1_A(x) = \infty$. So, (2.10) is proved.

COROLLARY. *Let $f \geqq 0$ be a l.s.c. function then*

$$\sum_{n=1}^{\infty} P^n f(x) = \{{}^{0}_{\infty} \tag{2.11}$$

except on a set of first category.

Proof. Denote $B_{kj} = \{P^k f \geqq 1/j\}$, $k = 0,\cdots, j = 1,2.\cdots$ $j \cdot P_k f \geqq 1_{B_{kj}}$. By Theorem 2.3 there exists a set of first category $E_{kj} \subset B_{kj}$ such that

$$x \in B_{kj} - E_{kn} \Rightarrow \sum_{n=1}^{\infty} P^n 1_{B_{kj}}(x) = \infty \Rightarrow \sum_{n=1}^{\infty} P^n f(x) = \infty.$$

Let $B = \bigcup_{k,j} B_{kj}$, $E = \bigcup_{k,j} E_{kj}$. E is a set of first category and

$$x \in B - E \Rightarrow \sum_{n=1}^{\infty} P^n f(x) = \infty$$

$$x \notin B \Rightarrow \sum_{n=1}^{\infty} P^n f(x) = 0.$$

THEOREM 2.4. *Let $f \geqq 0$ be a l.s.c. function, then*

$$Pf \leqq f \Rightarrow Pf = f \tag{2.12}$$

except on a set of first category.

Proof. Let $a_n < b_n$ be two positive rational numbers. If the set $N_n = \{f > b_n, Pf \leqq a_n\}$ is not nowhere dense, then there exists an open set $B \subset N_n$, because N_n is an intersection of the open set $\{f > b_n\}$ and the closed set $\{Pf \leqq a_n\}$, hence: $f - Pf \geqq (b_n - a_n)1_B$, thus: $(b_n - a_n) \sum_{k=1}^{K} P^n 1_B \leqq \sum_{k=1}^{K} P^k(f - Pf) = Pf - P^{k+1}f \leqq Pf$. Or $\sum_{k=1}^{\infty} P^k 1_B < \infty$ which contradicts Theorem 2.5. Hence

N_n is a nowhere dense set. Let $\{(a_n, b_n)\}$ be the sequence of all couples of rational numbers such that $b_n > a_n$. The set $N = \bigcup_{n=1}^{\infty} N_n$ is of first category and for each $x \notin N$ we have $Pf(x) = f(x)$, which proves the theorem.

COROLLARY. *Let $f \geqq 0$ be a continuous function then $Pf \leqq f \Rightarrow Pf = f$. In particular $P1 = 1$.*

Proof. Theorem 2.6 implies that $Pf = f$ on a dense set. But f and Pf are continuous functions, hence $Pf = f$ everywhere.

The following example shows that Theorems 2.3 and 2.4 are not true unless we assume the existence of an exceptional set of first category. This example is an improvement of an example given by Prof. S. R. Foguel.

EXAMPLE. Put $X = [0,1)$, let ϕ be the map of X into itself defined by $\phi(x) = \{3x\}$ ($3x$ minus the integral part of $3x$). One can describe X to be the unit circle and the transformation sends z to z^3 ($|z| = 1$), and then ϕ is a continuous map.

Let us define $Pf(x) = f(\phi(x))$. There exists a measure invariant under P that does not vanish on any open set (the Lebesgue measure) and thus $\sum_{n=1}^{\infty} P^n 1_A = \infty$ a.e. for every open set A. Theorem 2.3 implies that the process is conservative. If we put $x = a_1 a_2 a_3 \cdots$ ($x = \sum \frac{a_i}{3^i}$, $a_i = 0,1,2$) then $\phi(x) = a_2 a_3 \cdots$. Let B be the Cantor set: those x's such that $a_i \neq 1$, $i = 1,2,3,\cdots$ $\phi^{-n}(B) \supset B$ for each n, and $A = \bigcup_{i=1}^{\infty} \phi^{-n}(B) - B$ contains every fraction $x = a_1 a_2 a_3 \cdots$ where the digit 1 appears only finitely many times. The set A is of first category but it has the same cardinality as the continuum and it is dense in X. B^c is an open set but $\{0 < \sum_{n=0}^{\infty} P^n 1_{B^c} < \infty\} = A$, and $P1_{B^c} \leqq 1_{B^c}$ but $P1_{B^c} \neq 1_{B^c}$ and $\lim_{n \to \infty} P^n 1_{B^c} = 1_{(B \cup A)^c}$.

THEOREM 2.5. *Let $f \geqq 0$ be a l.s.c. subvariant function. Let $B = \{f > a\}$ then there exist an open set $\tilde{B}$ such that $\tilde{B} \supset B$, $P1_B \leqq 1_B$ and $\tilde{B} - B$ is a set of first category.*

Proof. Let us take a sequence $\{a_n\}$ such that $a_n \searrow a$, and consider the sets $B_n = \{f > a_n\}$, we have $B_n \subset B$. Define the functions $g_n = 1/a_n \min.(f, a_n)$, we have $g_n \geqq 1_{B_n}$ and $Pg_n \leqq g_n$, hence by Lemma 2.1 $i_{B_n} \leqq g_n$. But $x \notin B \rightarrow g_n(x) < 1$ for all n. Thus, by Theorem 2.2, there exists a set $N_n \subset B^C$ of first category so that $x \notin B \cup N_n \Rightarrow i_{B_n}(x) = 0$. We have $B_{n+1} \supset B_n$ and Lemma 2.1 implies $i_{B_{n+1}} \geqq i_{B_n}$. Similarly, $i_{B_n} \leqq i_B$, hence $\lim_{n \to \infty} i_{B_n} \leqq i_B$. On the other hand, for each

$x \in B$, $i_{B_n}(x) = 1$ for sufficiently large n, $i_{B_n}(x) = 1$, hence $\lim_{n\to\infty} i_{B_n}(x) = g(x) \geqq 1_B(x)$ and it is clear that $Pg \leqq g$. Thus by Lemma 2.1 we have $g \geqq i_B$, hence $\lim_{n\to\infty} i_{B_n} = i_B$. Denote $N = \bigcup_{n=1}^{\infty} N_n$. N is a set of first category and $x \notin B \cup N \Rightarrow i_{B_n}(x) = 0$ for all $n \Rightarrow i_B(x) = 0$. Let $\tilde{B}$ be the set $\tilde{B} = \{i_B > 0\}$, $\tilde{B} \subset B$ and $\tilde{B} - B \subset N$. Define the functions $h_n = n \cdot \min(i_B, 1/n)$. We have $Ph_n \leqq h_n$ and $\lim_{n\to\infty} h_n = 1_{\tilde{B}}$. Hence $P1_{\tilde{B}} \leqq 1_{\tilde{B}}$. which completes the proof of Theorem 2.7.

3. Ratio limit theorems. In this section we shall assume

ASSUMPTION 3.1. For all x and for every open set A we have

$$Pi_A(x) = 1 \tag{3.1}$$

LEMMA 3.1. *Let A be any open set. Then*

$$\sum_{n=1}^{\infty} P^n 1_A(x) = \infty \tag{3.2}$$

for all x.

Proof.

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} P^{n+k} 1_A(x) \geqq \lim_{k\to\infty} \sum_{n=0}^{\infty} P^k (I_{A^c} P)^n 1_A(x) = \lim_{k\to\infty} P^k i_A(x) = 1.$$

Thus

$$\sum_{n=0}^{\infty} P^n 1_A(x) = \infty.$$

Let $0 \leqq \beta \leqq 1$ be a function with a compact support E and the interior of the set $\{\beta = 1\}$ is non-empty. Let $\alpha = 1 - \beta$; define:

$$P_\beta = I_E \sum_{n=0}^{\infty} P(I_\alpha P)^n I_\beta .$$

In particular, if $\beta = 1_A$

$$P_A = I_A \sum_{n=0}^{\infty} P(I_{A^c} P)^n I_A . \tag{3.4}$$

From Lemma 3 of [4] we can conclude that P_β is a contraction on $\mathscr{B}(E, \Sigma_E)$.

The operator adjoint to P acts on the measures. It will also be denoted by P but will be written to the right of its variable. Thus

$$\mu P(A) = \langle \mu, P1_A \rangle = \int P1_A d\mu . \tag{3.5}$$

Equation (3.4) will occasionally be for σ-finite positive measures. A measure is said to be *invariant* if

$$\mu P = \mu. \tag{3.6}$$

In [4] it is proved that condition (3.1) implies that there exists a σ-finite invariant measure which is finite on compact sets and positive on open sets.

LEMMA 3.2. (a) *Let μ be a finite positive measure on E invariant under P_β then*

$$\lambda = \sum_{n=0}^{\infty} \mu(PI_\alpha)^n \tag{3.7}$$

is a σ-finite measure invariant under P, and $\lambda I_\beta = \mu$

(b) *Let λ be a measure invariant under P and $\lambda(E) < \infty$ then λI_β is invariant under P_β.*

Proof. (a) See [4]. Further,

$$\lambda I_\beta = \lambda PI_\beta = \sum_{n=0}^{\infty} \mu(PI_\alpha)^n PI_\beta = \mu I_E \sum_{n=0}^{\infty} (PI_\alpha)^n PI_\beta = \mu P_\beta = \mu.$$

(b) Assume $\lambda P = \lambda$, hence

$$\lambda I_\beta \sum_{n=0}^{N} (PI_\alpha)^n PI_\beta = \lambda \sum_{n=0}^{n} (PI_\alpha)^n PI_\beta - \lambda I_\alpha \sum_{n=0}^{N} (PI_\alpha)^n PI_\beta =$$

$$= \lambda \sum_{n=0}^{N} (PI_\alpha)^n PI_\beta - \lambda PI_\alpha \sum_{n=0}^{N} (PI_\alpha)^n PI_\beta = \lambda PI_\beta - \lambda (PI_\alpha)^{N+1} PI_\beta.$$

But from Lemma 3 of [4] we can conclude that

$$\lambda(PI_\alpha)^{N+1} PI_{\beta_{N\to\infty}} \to 0.$$

Thus

$$\lambda I_\beta P_\beta = \lambda I_\beta \sum_{n=0}^{\infty} (PI_\alpha)^n PI_\beta = \lambda PI_\beta = \lambda I_\beta.$$

LEMMA 3.3. *Let $f \geqq 0$ be a function subinvariant under P; then $I_E f$ is a subinvariant under P_β and $P_\beta I_E f \leqq I_E P f$.*

Proof. Let $f \geqq 0$ be a subinvariant under P. Hence

$$I_E \sum_{n=0}^{N} P(I_\alpha P)^n I_\beta f = I_E \sum_{n=0}^{N} P(I_\alpha P)^n f - I_E \sum_{n=0}^{N} P(I_\alpha P)^n I_\alpha f$$

$$\leqq I_E \sum_{n=0}^{N} P(I_\alpha P)^n f - I_E \sum_{n=0}^{N} P(I_\alpha P)^n I_\alpha P f = I_E P(I_\alpha P)^{N+1} f \leqq I_B P f \leqq I_B f.$$

Let N tend to ∞, hence $PI_Ef \leqq I_EPf \leqq I_Ef$.

Let λ be a σ-finite invariant measure. (X,Σ,λ,P) may be considered as a Markov process on the measure space (X,Σ,λ). (For the definition of such a process see [3]).

LEMMA 3.4. *The process (X,Σ,λ,P) is conservative.*

Proof. If the dissipative part of the process is non-empty then there exists a subinvariant function $f \geqq 0$ such that $Pf \neq f$. Let us take an open set A such that $\lambda(A) < \infty$, and hence $P_A 1_A = 1_A$, and $I_A Pf \neq I_A f$. Consider the process (A,Σ_A, mI_A, P_A); by [3] Chapter II it is a well defined Markov process. By Lemma 3.2 we have $P_A I_A f \leqq I_A P f \leqq I_A f \Rightarrow P_A I_A f \neq I_A f$. Hence the dissipative part of the process $(A,\Sigma_A,\lambda I_A,P_A)$ is non-empty, by Theorem A of [3] Chapter III. On the other hand, λI_A is a finite measure invariant under P_A, thus P_A is a conservative operator, a contradiction. Thus the process (X,Σ,λ,P) is conservative.

ASSUMPTION 3.2. There exists a unique invariant σ-finite measure λ, such that is finite on compact sets.

DEFINITION 3.1. A function f is said to be *almost continuous* if there exists an increasing sequence of continuous functions $\{f_n\}$ and a decreasing sequence of continuous functions $\{\bar{f}_n\}$ such that $\lim_{n\to\infty} f_n = \underline{f} \leqq f$ and $\lim_{n\to\infty} \bar{f}_n = \bar{f} \geqq f$, and $\lambda(\{\bar{f} > \underline{f}\}) = 0$

THEOREM 3.1. *Let f, g be two positive and almost continuous functions with compact supports. Let μ be any finite measure. Then:*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \langle \mu, P^n f\rangle}{\sum_{n=1}^{N} \langle \mu, P^n g\rangle} = \frac{\langle \lambda, f\rangle}{\langle \lambda, g\rangle}. \tag{3.8}$$

We shall first prove the following lemma:

LEMMA 3.5. *Let $0 \leqq \beta \leqq 1$ be a function with a compact support E and the interior of the set $\{\beta = 1\}$ is non-empty, then for each N and each $x \in X$, and every bounded function, $0 \leqq f$ we have*

$$\sum_{n=1}^{N} P^n I_\beta P_\beta f(x) \leqq \sum_{n=1}^{N} P^n I_\beta f(x) + \|f\|. \tag{3.9}$$

Proof. For each K we have

$$\sum_{n=1}^{N} P^n I_\beta \sum_{k=0}^{K} P(I_\alpha P)^k I_\beta f(x) = \sum_{n=1}^{N} P^n \sum_{k=0}^{K} P(I_\alpha P)^k I_\beta f(x) -$$

$$- \sum_{n=1}^{N} P^n I_\alpha \sum_{k=0}^{K} P(I_\alpha P)^k I_\beta f(x) = \Big(\sum_{n=1}^{N} P^{n+1} \sum_{k=0}^{K} (I_\alpha P)^k I_\beta f(x) -$$

$$- \sum_{n=1}^{N} P^n \sum_{k=0}^{K} (I_\alpha P)^k I_\beta f(x) \Big) = \Big(\sum_{n=0}^{N} P^n \sum_{k=0}^{K} (I_\alpha P)^k I_\beta f(x) -$$

$$- \sum_{n=1}^{N} P^n \sum_{k=0}^{K} (I_\alpha P)^{k+1} I_\beta f(x) \Big) = P^{N+1} \sum_{k=0}^{K} (I_\alpha P)^k I_\beta f(x) - P \sum_{k=0}^{K} (I_\alpha P)^k I_\beta f(x) +$$

$$+ \sum_{n=1}^{N} P^n I_\beta f(x) - \sum_{n=1}^{N} P^n (I_\alpha P)^{K+1} I_\beta f(x) \leqq \sum_{n=1}^{N} P^n I_\beta f(x) + P^{N+1} \sum_{k=0}^{K} (I_\alpha P)^k I_\beta f(x) \leqq$$

$$\leqq \sum_{n=1}^{N} P^n I_\beta f(x) + \| f \| \cdot P^{N+1} \sum_{k=0}^{K} (I_\alpha P)^k I_\beta 1(x) =$$

$$= \sum_{n=1}^{N} P^n I_\beta f(x) + \| f \| \quad P^{N+1} \sum_{k=0}^{K} (I_\alpha P)^k 1(x) - P^{N+1} \sum_{k=0}^{K} (I_\alpha P)^k I_\alpha P 1(x)) =$$

$$= \sum_{n=1}^{N} P^n I_\beta f(x) + \| f \| \cdot (P^{N+1} 1(x) - P^{N+1} (I_\alpha P)^{K+1} 1(x)) \leqq$$

$$\leqq \sum_{n=1}^{N} P^n I_\beta f(x) + \| f \| .$$

Hence

$$\sum_{n=1}^{N} P^n I_\beta \sum_{k=0}^{K} P(I_\alpha P)^k I_\beta f(x) \leqq \sum_{n=1}^{N} P^n I_\beta f(x) + \| f \| .$$

Let K tend to ∞, we get

$$\sum_{n=1}^{N} P^n I_\beta P_\beta f(x) \leqq \sum_{n=1}^{N} P^n I_\beta f(x) + \| f \| .$$

Proof of theorem 3.1. Let A be a compact set that contains the supports of the function f and g. Let $0 \leqq \beta \leqq 1$ be a continuous function with a compact support E and $A \subset \{\beta = 1\}$. There exists a sequence of integers $\{N_j\}$ such that the sequence

$$\frac{\sum_{n=1}^{N_j} \langle \mu, P^n f \rangle}{\sum_{n=1}^{N_j} \langle \mu, P^n \beta \rangle}$$

converges. In order to prove the theorem it is sufficient to show that

$$\lim_{j\to\infty} \frac{\sum_{n=1}^{N_j} \langle \mu, P^n f\rangle}{\sum_{n=1}^{N_j} \langle \mu, P^n \beta\rangle} = \frac{\langle \lambda, f\rangle}{\langle \lambda, \beta\rangle},$$

and to show also by a similar way that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \langle \mu, P^n g\rangle}{\sum_{n=1}^{N} \langle \mu, P^n \beta\rangle} = \frac{\langle \lambda, g\rangle}{\langle \lambda, \beta\rangle}.$$

Let us define a functional on $B(E,\Sigma_E)$:

$$\text{(3.10)} \qquad v(h) = \operatorname*{Lim}_j \left\{ \frac{\sum_{n=1}^{N_j} \langle \mu, P^n I_\beta h\rangle}{\sum_{n=1}^{N_j} \langle \mu, P_n \beta\rangle} \right\} \text{ (a Banach limit), } h \in B(E,\Sigma_E).$$

By Lemma 3.1 we have $\sum_{n=1}^{\infty} \langle \mu, P^n\beta\rangle = \infty$, hence Lemma 3.5 implies $v(P_\beta h) \leqq v(h)$. But $v(1_E) = 1$. Let $0 \leqq h \leqq 1_E$ then

$$v(P_\beta(1_E - h) = v(1_E - P_\beta h) = 1 - v(P_\beta h) \geqq 1 - v(h) = v(1_E - h).$$

Thus $v(P_\beta h) = v(h)$, and v is invariant under P_β. v is also defined as a functional on $C(E)$, but by Assumption 3.2 and Lemma 3.2 we have that there exists a unique functional on $C(E)$ which is invariant under P_β. This functional is the functional which is induced by the measure λI_β. Hence

$$h \in C(E) \Rightarrow v(h) = \frac{\langle \lambda I_\beta, h\rangle}{\langle \lambda I_\beta, 1_E\rangle} = \frac{\langle \lambda, I_\beta h\rangle}{\langle \lambda, \beta\rangle} \cdot f$$

is an almost continuous function. Let $\{f_n\}$ and $\{\bar{f}_n\}$ be the sequences of Definition 3.1. We have:

$$\frac{\langle I\lambda_\beta, f\rangle}{\langle \lambda, \beta\rangle} = \frac{\langle \lambda I_\beta, f\rangle}{\langle \lambda, \beta\rangle} = \lim_{n\to\infty} \frac{\langle \lambda I_\beta, f_n\rangle}{\langle \lambda, \beta\rangle} = \lim_{n\to\infty} v(f_n) \leqq v(f) \leqq v(f)$$

$$\frac{\langle \lambda I_\beta, f\rangle}{\langle \lambda, \beta\rangle} = \frac{\langle \lambda I,_\beta \bar{f}\rangle}{\langle \lambda, \beta\rangle} = \lim_{n\to\infty} \frac{\langle \lambda I_\beta, \bar{f}_n\rangle}{\langle \lambda, \beta\rangle} = \lim_{n\to\infty} v(\bar{f}_n) \geqq v(f).$$

Thus
$$v(f) = \frac{\langle \lambda I_\beta, f\rangle}{\langle \lambda, \beta\rangle} = \frac{\langle \lambda, f\rangle}{\langle \lambda, \beta\rangle}$$

i.e.
$$\lim_{j\to\infty} \frac{\sum_{n=1}^{N} \langle \mu, P^n f\rangle}{\sum_{n=1}^{N} \langle \mu, P^n \beta\rangle} = \frac{\langle \lambda, f\rangle}{\langle \lambda, \beta\rangle}.$$

COROLLARY 1. *Let f, g, be as in Theorem 3.1. Let μ be the Dirac measure δ_x then we have*

$$\frac{\sum_{n=1}^{N} P^n f(x)}{\sum_{n=1}^{N} P^n g(x)} = \frac{\langle \lambda, f\rangle}{\langle \lambda, f\rangle}. \tag{3.11}$$

COROLLARY 2. *Let A, B be condtionally compact sets, such that $\lambda(A) > 0$ $\lambda(B) > 0$ and $\lambda(\partial A) = \lambda(\partial B) = 0$ (the boundaries); then 1_A and 1_B are almost continuous functions. Let μ be any finite measure. Then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \mu P^n(B)}{\sum_{n=1}^{N} \mu P^n(A)} = \frac{\lambda(B)}{\lambda(A)} \tag{3.12}$$

COROLLARY 3. *Let A, B be as in Corollary 2. Let μ be the Dirac measure δ_x then*

$$\lim_{n\to\infty} \frac{\sum_{n=1}^{N} P^n(x,B)}{\sum_{n=1}^{N} P^n(x,A)} = \frac{\lambda(B)}{\lambda(A)}. \tag{3.13}$$

DEFINITION 3.2. The process is said to be a Harris process if

$$\lambda(A) > 0 \Rightarrow i_A(x) = 1 \quad \text{for all } x \in X. \tag{3.14}$$

The following lemma is well known, we shall give its proof for completeness.

LEMMA 3.6. *If the process is a Harris process then Assumption 3.2 is valid.*

Proof. Let μ be any invariant measure. It can be decomposed into $\mu = \mu_1 + \mu_2$ where $\mu_1 \prec\!\!\!< \lambda$ and $\mu_2 \perp \lambda$. Let:

$$0 = \lambda(A) = \lambda P(A) = \langle \lambda, P1_A\rangle \Rightarrow \lambda(\{P1_A > 0\}) = 0 \Rightarrow$$

$$\Rightarrow \mu_1(\{P1_A > 0\}) = 0 \Rightarrow \langle \mu_1, P1_A\rangle = \mu_1 P(A) = 0 \Rightarrow \mu_1 P \prec\!\!\!< \lambda.$$

Consider the process (X,Σ,λ,P), (3.14) implies that the process is ergodic and by Chapter VI of [3] Theorem A, there can exist at most one σ-finite invariant measure for the process (X,Σ,λ,P). Hence $\mu_1 = a\cdot\lambda$. We shall prove that $\mu_2 = 0$. There exists a set A such that $\lambda(A) > 0$ but $\mu_2(A) = 0$. Denote $A_n = \{P^n 1_A > 0\}$. We have for each n: $0 = \mu_2(A) = \mu_2 P^n(A) = \langle \mu, P^n 1_A\rangle \Rightarrow \mu_2(\bigcup_{n=1}^{\infty} A_n) = 0$. But by Lemma 2.2 and (3.14) we have $\bigcup_{n=1}^{\infty} A_n = X$. Thus $\mu_2 = 0$.

The following example will show that Harris' condition (3.14) is not necessary for assumption 3.2 to hold.

Example. Let $X = R^1$ the real line, λ the Lebesgue measure, μ a probability measure which support is not contained in a discrete subgroup of R^1, but the support of μ is a countable set and hence $\lambda \perp \mu$. Let us also assume $\int x\mu(dx) = 0$. Define $Pf(x) = \int f(x-y)\mu(dy)$. By [2] Chapter VI, 10, if A is an open set then every $x \in R^1$ enters A some time with probability 1, i.e. $Pi_A(x) = 1$. Hence assumption 3.1 is satisfied. But this process is not a Harris process. Consider the measure $\tilde{\mu} = \sum_{n=0}^{\infty} \frac{\mu P^n}{2^{n+1}}$, let F be its support. It is obvious that F is a countable set. Hence $\lambda(F) = 0$. Let $F_x = F + x$ then $\lambda(F_x) = 0$. But $P^n(x,F_x) = 1$ for al. n or $i_{F_x^c(x)} = 0$, i.e., (3.14) is not satisfied.

On the other hand Assumption 3.2 holds. It is clear that $\lambda P = \lambda$; we shall prove that if ν is a σ-finite invariant measure which is finite on compact sets that $\nu = a\cdot\lambda$. Let A be an open bounded interval:

$$\nu(A) = \nu P(A) = \langle \nu, P1_A\rangle = \int\int 1_A(x-y)\mu(dy)\nu(dx) =$$
$$= \int\int 1_A(x-y)\nu(dx)\mu(dy) = \int \nu(A+y)\,\mu(dy).$$

Define: $f(x) = \nu(A - x)$, it is easy to see that $f(x)$ is a continuous function, and we have $Pf = f$. Define $f_N = \min(f,N)$; in order to prove that f is constant it is sufficient to prove that f_N is constant for each N. But $Pf_N \leqq f_N$ and by the Corollary to Theorem 2.6 we have $Pf_N = f_N$. If f_N is not a constant then there exists $a < b$ such that $B = \{f_N > b\}$ and $E = \{f_N < a\}$ are non-empty open sets, and $B \cap E = \varnothing$. But by Theorem 2.5 there exists an open set B such that $B \subset \tilde{B}$ and $\tilde{B} - B$ is a set of first category, and hence $\tilde{B} \cap E = \varnothing$, such that $P1_{\tilde{B}} \leqq 1_{\tilde{B}}$. Thus, $x \in E \Rightarrow P^n(x,\tilde{B}) = 0$ for all n, which contradicts Assumption 3.1. Hence f is a constant, i.e. $\nu(A - x) = \nu(A)$ for every open interval A. Thus ν is the Haar measurement on R^1, i.e. $\nu = a\cdot\lambda$.

References

1. K. L. Chung, *The general theory of Markov processes according to Doeblin*, Z. Wahrscheinlichkeitstheorie verw. Geb. **2** (1964), 230–254.

2. W. Feller, *An introduction to probability theory and its applications*, Vol. II, John Wiley and Sons, Inc., 1966.

3. S. R. Foguel, *The ergodic theory of Markov processes*, to appear.

4. S. R. Foguel, *Existence of a σ-finite invariant measure for a Markov process on a locally compact space*, Israel J. Math. **6** (1968) 1–4.

5. S. R. Foguel, *Ergodic decomposition of a topological space*, to appear in Israel J. Math.